9. THE INEFFABLE

§9.1. Proofs Of The Existence of God

We've finished the mathematical content of this book. This final chapter consists of some philosophical/theological musings that arise in some minds as a result of encountering those parts of mathematics that deal with the edge of the rational universe. If you have no interest in the fundamental questions of life then it's best that you skip this chapter.

Now I wish to make it clear that my purpose in writing these notes is to communicate what I see as the nature of mathematics, not to talk about religion. I once received an email from an angry reader who believed that this chapter was the 'pill' and the previous ones were the 'sugar coating' and that my whole aim was to sneak religion in under the radar.

If you are one of those who get angry at the mere mention of anything religious then you'd better not read on. But let me emphasise that I am not here talking about the ineffability of God, but the ineffability of mathematics. I present some 'proofs' of the existence of God, merely as a vehicle for discussing bad, or erroneous, logic. But, by the same token, the fact that I expose such faulty reasoning doesn't mean that I am arguing *against* the existence of God.

I lay no claim to any professional expertise in either philosophy or theology. But I can't help going beyond the mathematics of the ineffable to the ineffable itself. The word 'ineffable' means 'inexpressible in words'. It's a word that not only appears in hymns, describing God, but also in a large number of nineteenth century novels. We don't use the word today, yet there's as much interest in the transcendental as ever.

There's a fundamental contradiction in the desire to discuss the ineffable – to say something meaningful about something that can't be expressed in words. But by a little distortion of the meaning we can think of the ineffable as that which transcends logic.

Can one prove that God exists? There have been many attempts over the centuries. A very simplistic argument, at least in the Christian tradition, runs as follows. The Bible says that God exists. The Bible says that everything in the Bible is the word of God and so must be true. Therefore God exists. Put more simply it says “GOD EXISTS AND THIS STATEMENT IS TRUE”.

One need not spend too much time in refuting this feeble argument. Just one word is needed – self-referentiality. It would have been far better if the Bible had made just two claims:

verse 1: God exists.	**verse 2:** Everything in this Bible is false.

If verse 2 is true then both verse 1 and verse 2 are false. But this leads to a contradiction. Oh well then, verse 2 must be false. So it is false that everything in this little Bible is false. So something must be true. It can’t be verse 2 because we’re assuming that it’s false. It must therefore be verse 1 that’s true. Therefore God exists!

This might seem momentarily convincing until we realise that any statement could have taken the place of ‘God exists’ – even ‘God does not exist’. The problem lies in the fact that verse 2 is ‘self-referential’ – it refers to itself.

One must refrain from considering any sentence that refers to its own truth. Such self-referential statements are meaningless and meaningless statements used in a logical argument can lead to paradoxes such as the above.

I remember, when training for my accreditation as a lay preacher many years ago, that I had to study many of these arguments – mostly with big names like ‘the ontological argument’. I mostly forget what they were.

One of the ones I do remember went like this. We define God to be a being that’s perfect in every way. Now existence is more perfect that non-existence. So if God didn’t exist this would contradict our definition. Therefore God exists.

The problem with this argument is that it assumes the existence of a being that is perfect in every way but who does not exist. The contradiction comes from assuming simultaneously the existence and the non-existence and has nothing to do with perfection.

We might define ‘infinity’ as “a number that’s bigger than every number” and ask the question, “Does this infinity exist?” Well a number that doesn’t exist can’t be bigger than every number. (In fact a non-existent number can't be bigger than *any* number.) Therefore infinity must exist. But, of course, such an 'infinity' must be bigger than itself, a situation that is clearly untenable.

The explanation for this paradox is that 'not existing' is not a property of something. It is the absence of something with a given property. We could say that a non-existent number can't be even. But nor can it be odd. The statement '*n* is even' is not true or false of a non-existent number – it is meaningless. In the same way 'God is perfect' is not true or false if it is the case that God does not exist.

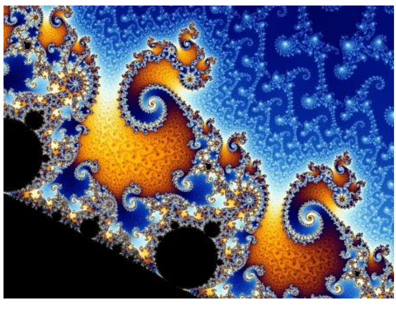
§9.2. Proof by Design

Another proof that God exists, one that was very popular in Victorian times, is Proof by Design. The world is a complex, finely balanced precision structure. If certain parameters were changed by only a small amount life would not be possible. It could not possibly have arisen by chance. There must have been an intelligent Creator. A watch could not come about by cogs, and other components, just throwing themselves together. So the universe must have been created by a Divine Clockmaker.

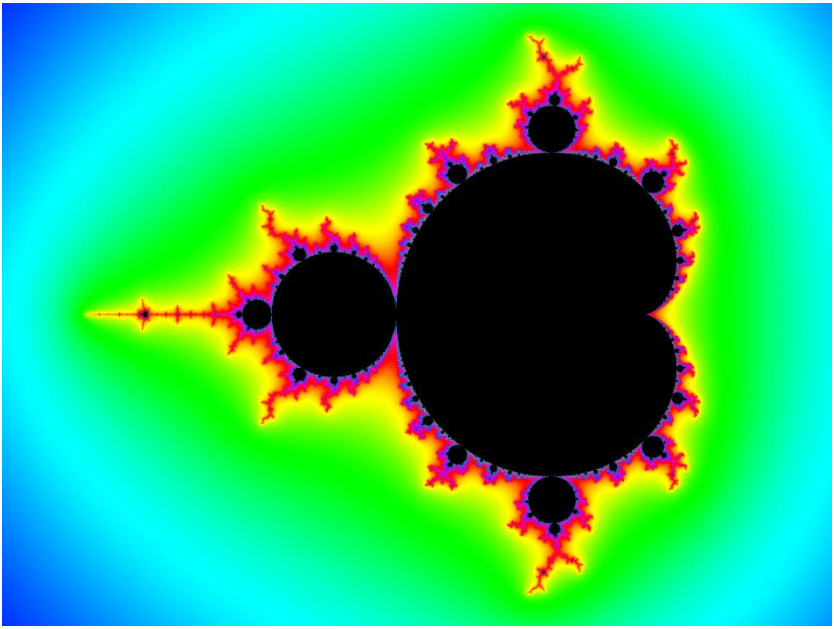
But then along came Darwin and his Theory of Evolution. Then came chaos theory, and the theory of fractals and complexity. It *is* possible for complexity to arise from simple rules. This can occur in biology, with the amazingly complex variety of plant structure arising from a small number of biological rules. If God created flowers he didn't do it the way an artist might painstakingly paint a complicated picture. You can't argue if something is extremely complicated it must have been the result of an extremely brilliant maker. On the other hand you could argue, and many do, that to create life by a process of evolution that achieves complexity from a small set of simple rules is so brilliant that only a supreme mind could have thought of it.

§9.3. The Mandelbrot Set

You'll have to make up your own mind on this. Let me simply describe the most well-known example of complexity arising from simple rules. Perhaps you have seen pictures of the Mandelbrot set. It's a design, best seen in colour, which is startlingly beautiful. There is a boring bit in the middle, usually coloured black, and an outside that's almost as boring. It's the region between the two that is amazingly complex. If you zoom in to a part of the Mandelbrot set in this intermediate area you'll see complexity upon complexity. The more you zoom in, the more fascinating the pattern becomes. Parts of the pattern, when you zoom in, look very much like other parts at a lower magnification, but they're subtly different. Here's a view of a small portion of the pattern.



Here's the whole image, or at least all of the region that contains the interesting stuff.



This pattern is constructed by a computer program using a few very simple rules. The program takes one point at a time on the 'canvas' and works out what colour to make it. It's a very tedious process, and one would only attempt to do it with a computer program. You have to consider every point on the rectangular canvas. Well, actually there are infinitely many points in a given rectangle so you choose a certain resolution and consider the individual pixels on a computer screen.

You choose at the outset, along with the scale, certain numbers N and R. You might, for example, choose N = 100 and R = 10. The number N represents the maximum number of steps you'll perform for each point and R represents the radius of a circle whose centre is at the middle of the screen.

Each point is considered in turn. You move systematically so that at the end you'll have considered every pixel on the screen. A certain calculation is carried out to determine what colour that pixel should be coloured and the image is built up in this way.

This calculation generates a sequence of points, though these points are not plotted. The sequence starts at the point whose colour we are determining. There's a very simple rule that calculates the next point in the sequence.

If a point in the sequence stays within the circle, of radius R, for N steps we colour that starting point black. If it breaks out of the circle we colour that original point some colour, depending on how long it took the sequence to 'escape' from the circle The points in the sequence don't get plotted, only the starting point for each step.

You decide on a certain palette of colours. You might choose red if the sequence breaks out after within 10 steps, orange if it takes up to 20 steps, blue if it stays within the circle for 30 steps before escaping and so on. These decisions on colours, as well as the choice of N and

R, may make your picture look different to mine, but the overall effect will be much the same.

A black and white version is far less interesting but still displays the enormous complexity of the Mandelbrot set. In this case, if the sequence breaks out of the circle after N steps it's coloured white. As with the coloured version, points whose sequence remains within the circle for N steps, and quite probably will remain inside forever, are coloured black.

All that remains is to tell you how you go from one point to the next in generating each sequence.

If you're not frightened by a little bit of algebra, here's the simple rule to move a point one step to create the next point in each sequence. The centre of the canvas is the origin for the x-y plane, the point to which measurements of all other points are referred.

A horizontal axis through the origin is called the x-axis and the vertical axis through the origin is called the y-axis. A point has coordinates (x, y) if it is x units to the right of the origin and y units up. If x or y is negative then the point is to the left, or below (or both) of the origin.

The rule for getting the next point (X, Y), following the point (x, y) in the sequence, is:

$$\left.\begin{array}{l} X = x^2 - y^2 + a \\ Y = 2xy + b \end{array}\right\}.$$

where (a, b) is the point whose colour we're trying to find. We begin with (x, y) = (a, b).
(If you know about complex numbers these equations can be expressed even more simply as Z = $z^2 + a$.)

There are websites on the internet where you can see the main image and where you can zoom in on a particular region, just like in Google Maps.

Let's do a few calculations to give you an idea. I'll just do the black and white version, using N = 10 and R = 10.

I'll generate the sequence starting with the point (0, 1) to see whether to colour it black or white. I begin as follows:

a	***b***	***x***	***y***	**X** $x^2 - y^2 + a$	**Y** $2xy + b$	**r** $\sqrt{X^2 + Y^2}$
0	1	0	1	–1	1	1.414
0	1	–1	1	0	–1	1.000
0	1	0	–1	–1	1	1.414

The a and b columns remain the same throughout. The x and y columns begin with a copy of the a and b columns. Then we calculate the last three columns and transfer the results to the x and y columns in the next row.

In this case there's no need to go any further. Clearly if the sequence repeats it must continue to repeat and so the points will stay within the circle of radius 10 forever. We should colour the point (0, 1) black.

Let's now try (1, 1).

a	***b***	***x***	***y***	**X** $x^2 - y^2 + a$	**Y** $2xy + b$	**r** $\sqrt{X^2 + Y^2}$
1	1	1	1	1	3	3.162
1	1	1	3	–7	7	9.899
1	1	–7	7	1	–97	97.005

We stop because the sequence has escaped. The point (1, –97) is outside of the circle with radius 10. So we colour (1, 1) white.

Try (– 0.5, 0.5).

a	***b***	***x***	***y***	**X** $x^2 - y^2 + a$	**Y** $2xy + b$	**r** $\sqrt{X^2 + Y^2}$
–0.5	0.5	–0.5	0.5	–0.5	0	0.5
–0.5	0.5	–0.5	0	– 0.25	0.5	0.559
–0.5	0.5	– 0.25	0.5	– 0.688	0.25	0.732
–0.5	0.5	– 0.688	0.25	– 0.090	0.1562	0.1802
–0.5	0.5	– 0.090	0.156	– 0.516	0.4719	0.700
–0.5	0.5	– 0.516	0.472	– 0.456	0.013	0.456
–0.5	0.5	– 0.456	0.013	– 0.292	0.489	0.569
–0.5	0.5	– 0.292	0.488	– 0.653	0.215	0.688
–0.5	0.5	– 0.653	0.215	– 0.119	0.220	0.250
–0.5	0.5	– 0.119	0.220	– 0.534	0.448	0.697

After 10 steps we are still inside the circle of radius 10, so (– 0.5, 0.5) should be coloured black.

Variations to the values of N and R don't affect the picture much. If we'd used N = 100 and R = 100 instead this will only have affected a few pixels on the border.

So great complexity can arise from very simple rules. Proof By Design is not a good proof that God exists.

§9.4. Free Will

A fundamental prerequisite to having a religious faith is a belief in free will. If you don't have the freedom to choose to accept or reject the belief, what's the point? Mind you, this hasn't stopped people in some parts of the world forcing others to accept their faith at gunpoint.

A common argument against religion is to claim that you have discovered some psychological or biological cause for religious belief. Oppressed people, who have a miserable life, will believe in an after-life where pain and suffering and poverty will be no more. It's just wishful thinking.

Others claim to have discovered a God-gene, which explains why some of us have a religious belief and others don't. Still others claim that all emotions and all thought is purely a result of biochemical processes. The human brain is influenced by its biochemistry and your belief in

God can be explained by your diet, or your genetic makeup.

It's an interesting thought that a belief in a purely deterministic universe is a contradiction. If I assert that my thoughts are determined by the laws of physics, chemistry and biochemistry, then my assertion is also the result of such deterministic processes.

In what sense could such a belief have any validity? The concept of truth presupposes that there's something beyond the material world. Otherwise what we might call true statements are just the babble of the mechanistic automata we call human beings.

But to say that there's something beyond the material world is a long way short of believing any sort of God. It is intellectually respectable to be an atheist, but I fail to see how it could be considered intellectually respectable to believe that human beings are purely machines, with no free will. Of course I can believe that *my* thoughts are valid truths, while *yours* are the result of deterministic processes. It's an interesting idea that I'm the only reality, complete with free will, and the rest of the universe is simply an illusion. I don't believe that, but I can't think of a way of proving to you that I have consciousness.

Whatever we might claim to believe we all act as if we have free will. It might be a great illusion but if it is we’ll never know it. But of course, as we all know, free will is impacted upon by all sorts of external forces, and even internal, biochemical ones. No-one can claim to be completely free.

In *The Age of Reason* by Jean-Paul Sartre, the protagonist wants to be completely free. As a result he refuses to make any commitments, because that would limit his freedom. “If I marry her I’ll remove, or at least reduce, my chance of marrying someone else.” Every decision involves a certain reduction in freedom. Better not make any decisions.

So the person who’s so determined to maximise his free will is forced to lock himself into a prison of indecision. He ends up with *less* freedom. Free will is a currency that must be spent or it becomes worthless.

I’m now going to describe a demonstration that purports to prove that people don’t have free will, even in a situation where there appear to be no external forces. Most people believe that they’re completely free when they select their lotto numbers, although certain combinations get chosen less frequently than others because people believe that they’re not random enough.

Would you choose the numbers 1, 2, 3, 4, 5, 6? It’s just as likely to come up as a more random sounding choice. The concept of random numbers is another area where there's an interface between mathematics and philosophy, but I'll not pursue this here.

This demonstration is designed to be performed in public, but you can just read about it and think about it. You have an audience from which you select five volunteers. You ask them to stand out the front in a line. Then you introduce the theme of free will as follows.

"Most people believe that they have free will. On being asked a certain question we might be influenced by certain facts, but if we have no facts, such as when choosing lotto numbers, we believe we can freely make up our minds." Check that your five volunteers agree with this. If any say no, you'd better replace them.

"I'm about to give you each a card that asks a question about one of the other volunteers. The question won't even identify who that person is. At this stage they'll only be identified as person A, person B, and so on. You must freely choose an answer, YES or NO. It doesn't matter whether you're correct, or not, because after all you don't know yet to whom the question is referring. Oh, and you mustn't let anyone else know your question."

You give them cards, each of which has the same type of question:

Will person A give the correct answer to their question?

except that each person's question will refer to a different person: A, B, C, D, E.

You give each volunteer a card. On one side it reads YES and on the other side it reads NO, with these answers written large enough that the audience can read it. "Now I want you to display your answer by turning your card so everyone can see that answer."

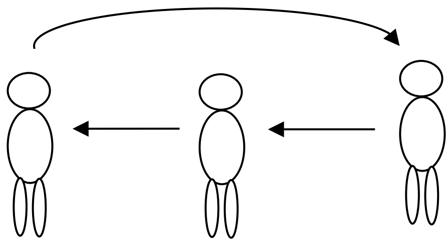
YES **NO**

Your volunteers will display a sequence of YES's and NO's. Perhaps they will all be YES, or all NO. You must select three of the volunteers so that you have an **even number** of NO's. Tell the other two to sit down. Do this as follows:

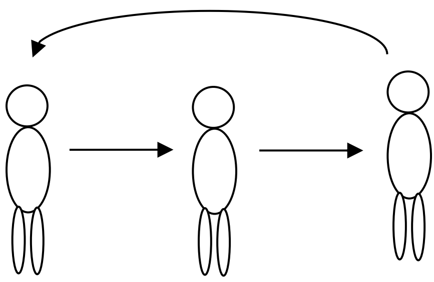
# who say NO?	**0**	**1**	**2**	**3**	**4**
CHOOSE	YES YES YES	YES YES YES	YES YES YES	NO NO YES	NO NO YES

You'll notice that I haven't included the case where they all say "NO". In this case it's not possible to choose an even number of NO's. Luckily the chance of this happening should be only 1 in 32. The demonstration I have in mind won't work in this case. I'll discuss later what you might do when this happens.

Ask each of the three to display their question as well as their answer. You now announce who the questions refer to. It will look as if you have determined this in advance, but you choose people only at this time. You'll only choose someone who is in your chosen three, and never themselves. In fact, for best effect, your choice should be as follows, where the arrow shows who their question refers to.



Now ask the two people on the right hand to swap places so that the people referred to are as shown.



It doesn't really matter who is referring to whom, but it will seem a little less artificial if an adjustment is made. Now you ask any one of the three, "did you feel that you were completely free to choose your answer? You didn't feel constrained in any way?" They of course will insist that they were free.

"But I will show you that you were *forced* to choose as you did because of the answers of the others. If you had chosen otherwise there would have been a logical contradiction. Suppose you had chosen otherwise."

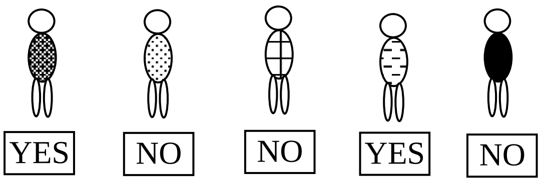
At this you instruct them to turn their card over. You then say, "suppose you were correct." Give them a card marked with a tick, signifying that their answer was correct. Hopefully they can manage to hold all three cards without dropping any!

√	X
CORRECT	**INCORRECT**

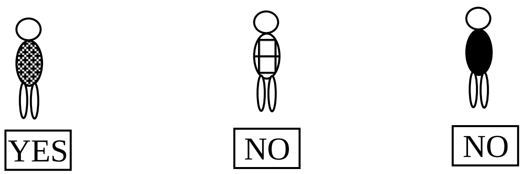
You then infer whether the person that their question relates to is correct or incorrect. You then give them a card with the appropriate word. Then you do this again so that the third person is holding a tick/cross card. Then you do this one more time so that you determine whether the first person is correct or incorrect. Because of the odd number of NO's this will conflict with the card they are holding, and so you will have reached a contradiction.

“See, if you had answered differently to the way you did, your answer must have been incorrect, because a correct answer would lead to a contradiction. You then take away all the tick and cross cards and start all over again, this time giving the cross card to that person to display. You repeat the whole process, and discover that *again* you get a contradiction! “See, if you’d chosen other than what you did there would have been a logical contradiction. This proves that you *had* to choose the way you did!

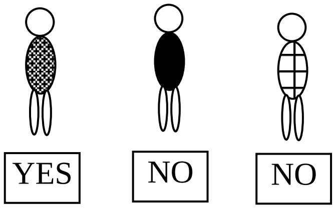
Here’s an example of how this might work out. Remember each person’s question refers to the person on *our* right, except the last, whose question refers to the first.
Suppose the five answers are as follows.



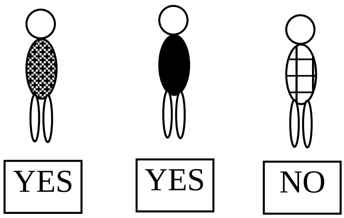
We choose two NO’s and a YES as follows.



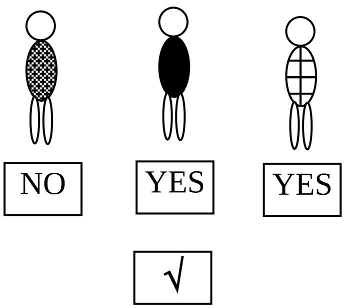
Now we swap the last two and move them together.



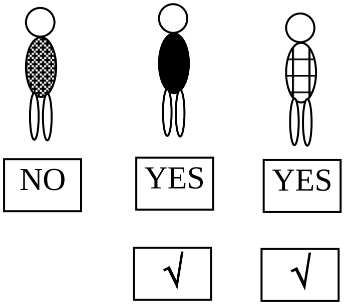
We get the middle person (it could have been any of them) to change his answer to see what logical implications this might have.



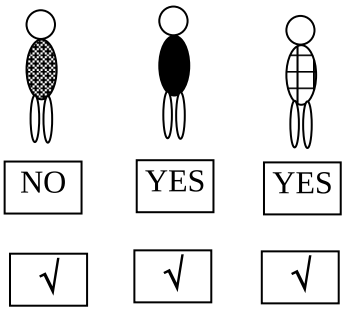
We suppose that the middle person was correct and you gave him a card containing a √.



The middle person said that the person on his left was correct (YES) and he was correct (√) so she must be correct. We give her a tick.

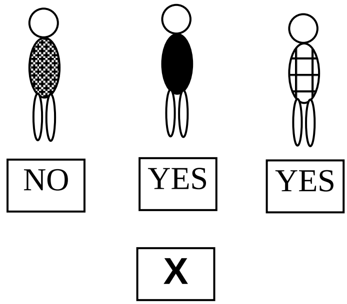


The third person says that the first would be correct, and the third person has a tick, so he is right. The first person gets a tick.

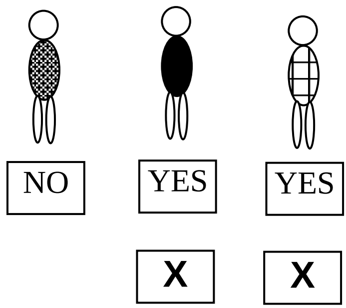


But the first person says that the middle person will say NO and he is supposed to be correct in saying this (we have had to give him a tick). But the middle person says YES. This is a contradiction.

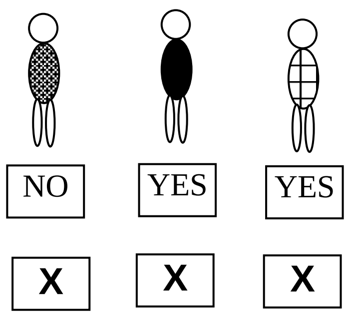
Now suppose the middle person, if he had said YES, would have been wrong. Give him a card with an X. We carry out the whole process afresh and again get a contradiction.



The middle person said that the person on his left was correct (YES) but he was wrong (X) so she must be incorrect. We give her a cross.



The third person says that the first will be correct and she is wrong, so he will be incorrect. Give the first person a cross.



But the first person says that the middle person will *not* be correct, and he is incorrect in saying this, so the middle person is correct. But the middle person is incorrect!

There's a problem if all of the original five people say NO. We'll not be able to select three people with an even number of NOs. This situation would be quite rare, but if it does arise you have the following options.

(1) Confess that the demonstration didn't work.

(2) Upbraid them for not having enough faith in their fellow volunteers and choose a fresh set of volunteers.

(3) Go through the above analysis to show that you get a contradiction, no matter whether the first person is correct or incorrect. Ask them how they have managed to defy logic and leave it at that. Hopefully nobody will ask "what has this got to do with free will?"

Remember that as you go round the circle of three (or all five) if someone has said YES then the next person will get the same tick or cross as they have, but every time you strike a NO the ticks and crosses swap over.

Clearly with an odd number of such swaps around a circle with an odd number of people, there's bound to be a contradiction. Of course this only superficially has something to do with free will! It merely demonstrates the fact that the questions are indirectly self-referential.

So, after all this, does God exist? You certainly won’t find the answer in this book or even the Bible, as useful as the Bible has been to many people. If there is a God and he chooses to reveal himself to you, then you’ll know. Otherwise you have the free will to use the Axiom of Choice to not believe in God. (Actually that’s not quite what the Axiom of Choice says but never mind.)

Oh, you don’t believe in the Axiom of Choice. That’s a logically valid position to take. What? You don’t even believe in free will. Then why are you interested in proving things at all. You’re simply a pre-programmed robot.

But let me remind you that this is a mathematics book, not a religious one. My goal is to explain the fact that logic has its limitations. As a mathematician I’m a great believer in it, but I’m fascinated to discover that there are impossible, uncountable, undecidable, unprovable things out there at the edge of the rational universe.

As the great bard once said:

There are more things in heaven and earth, Horatio,
Than are dreamt of in your philosophy.

[Hamlet Act 1, Scene 5]

THE MATHEMATICIAN'S CREED

I believe in the validity of standard logic,
Provided there are no infinite chains of referentiality.
I believe that Mathematics was created from the empty set
And that the ZF axioms are consistent.

I believe in Mathematical Intuition,
Informed by rigorous proof,
But inspired by Mathematical Imagination
Fuelled by countless cups of coffee.

I believe in the Axiom of Choice
And the Continuum Hypothesis
And whatever other axioms I might find convenient to use
Provided they have been proved consistent with ZF.

I believe that Mathematics contains no facts
But depends on definitions and sets of axioms
I believe that mathematics is independent of the material world
So that it can be understood by a disembodied angel.

Yet I believe that mathematics is the one great tool
For understanding the material world,
It guides and underpins the Kingdom of Science
And has brought great benefit to mankind.

I believe that great as Mathematics is
There are truths that lie beyond its reach.
Our minds can soar into realms unknown
But more truth lies beyond the edge of the rational universe.